Painlevé Analysis and Symbolic Computation for a Nonlinear Schrödinger Equation with Dissipative Perturbations

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The nonlinear Schrödinger equations with small dissipative perturbations are of current importance in modeling weakly nonlinear dispersive media with dissipation. In this paper, the Painlevé formulation with symbolic computation is presented for one of those equations. An auto-Bäcklund transformation and some exact solutions are explicitly constructed.

Key words: Nonlinear Schrödinger equation with dissipative perturbations, Symbolic computation, Painlevé analysis, Bäcklund transformation, Mathematical methods in physics.

The nonlinear Schrödinger equations with small dissipative perturbations are powerful tools in modeling weakly nonlinear dispersive media with dissipation. For instance, they occur in plasma physics to describe the interaction of the Langmuir and ion-sound waves with small velocities, and in hydrodynamics as a generalized Ginzburg-Landau equation to study the instability of Couette-Taylor, Poiseuille and plane-parallel flows [1–6]. In this paper, the Painlevé analysis will be combined with symbolic computation to analytically solve one of those equations, written as

$$i u_t + (1 - i\alpha) u_{xx} + (-2 + i\beta) u |u|^2 + i\gamma u = 0, \tag{1}$$

where α, β and γ are in general non-negative dissipation coefficients, while $\gamma < 0$ describes linear pumping, as seen in [5].

We shall assume that

$$\alpha > 0 \quad \text{and} \quad \gamma \neq 0, \tag{2}$$

since the special case $\alpha = 0$ of the nonsoliton problem can be found in [7, 8], and the linear instability threshold at $\gamma = 0$ has been addressed as well [6].

We re-write (1) as the coupled system

$$i u_t + (1 - i\alpha) u_{xx} + (-2 + i\beta) u^2 v + i\gamma u = 0, \tag{3}$$

$$-i v_t + (1 + i\alpha) v_{xx} + (-2 - i\beta) u v^2 - i\gamma v = 0, \tag{4}$$

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where $v = u^*$ is the complex conjugate of u, so as to remove the non-analytic nature of (1) and make it amendable to the Painlevé expansion [9, 10]

$$u(x,t) = \phi^{-\varepsilon}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\, \phi^j(x,t),$$

$$v(x,t) = \phi^{-\eta}(x,t) \sum_{j=0}^{\infty} v_j(x,t)\, \phi^j(x,t) \tag{5}$$

around a non-characteristic movable singular manifold given by an analytic function $\phi = 0$, where the functions $u_j(x,t)$ and $v_j(x,t)$ are also analytic, while ε and η could be real or complex numbers.

The leading-order analysis is done for

$$u \sim u_0 \phi^{-\varepsilon}, \quad v \sim v_0 \phi^{-\eta}, \tag{6}$$

by which (3) and (4) yield

$$\varepsilon + \eta = 2 \tag{7}$$

and

$$\phi^{-\varepsilon-2}\colon\ \varepsilon(\varepsilon+1)(1 - i\alpha) u_0 \phi_x^2 + (-2 + i\beta) u_0^2 v_0 = 0, \tag{8}$$

$$\phi^{-\eta-2}\colon\ \eta(\eta+1)(1 + i\alpha) v_0 \phi_x^2 + (-2 - i\beta) u_0 v_0^2 = 0. \tag{9}$$

When ε and η are both real, we get from (7), (8) and (9)

$$\varepsilon = 1, \quad \eta = 1, \tag{10}$$

$$\beta = 2\alpha \quad \text{and} \quad u_0 v_0 = \phi_x^2. \tag{11}$$

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Equations (5) are then applied to (3) and (4) to get the recursion relations,

$$\begin{pmatrix} (j-1)(j-2)(1-i\alpha)\phi_x^2+2(-2+i\beta)u_0v_0 & (-2+i\beta)u_0^2 \\ (-2-i\beta)v_0^2 & (j-1)(j-2)(1+i\alpha)\phi_x^2+2(-2-i\beta)u_0v_0 \end{pmatrix}\begin{pmatrix} u_j \\ v_j \end{pmatrix}=\begin{pmatrix} U_j \\ V_j \end{pmatrix}, \tag{12}$$

where $j\geq 1$, $U_j=U_j(\phi,u_1,\dots,u_{j-1},v_1,\dots,v_{j-1})$ and $V_j=V_j(\phi,u_1,\dots,u_{j-1},v_1,\dots,v_{j-1})$.

It is seen that the resonances occur at $j=-1$, 0, 3 and 4, of which $j=-1$ and $j=0$ correspond to the fact that ϕ, u_0 and v_0 are arbitrary. We write down the recursion relations one by one:

$$j=0\text{: Equations (11),} \tag{13}$$

$$\begin{aligned} j=1\text{:}\quad & \begin{pmatrix} 2(-2+i\beta)u_0v_0 & (-2+i\beta)u_0^2 \\ (-2-i\beta)v_0^2 & 2(-2-i\beta)u_0v_0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\ & =\begin{pmatrix} -iu_0\phi_t-2(1-i\alpha)\phi_x u_{0,x}-(1-i\alpha)u_0\phi_{xx} \\ iv_0\phi_t-2(1+i\alpha)\phi_x v_{0,x}-(1+i\alpha)v_0\phi_{xx} \end{pmatrix}, \end{aligned} \tag{14}$$

$$\begin{aligned} j=2\text{:}\quad & \begin{pmatrix} 2(-2+i\beta)u_0v_0 & (-2+i\beta)u_0^2 \\ (-2-i\beta)v_0^2 & 2(-2-i\beta)u_0v_0 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \\ & =\begin{pmatrix} iu_{0,t}+(1-i\alpha)u_{0,xx}+2(-2+i\beta)u_0u_1v_1+(-2+i\beta)u_1^2v_0+i\gamma u_0 \\ -iv_{0,t}+(1+i\alpha)v_{0,xx}+2(-2-i\beta)v_0u_1v_1+(-2-i\beta)v_1^2u_0-i\gamma v_0 \end{pmatrix}, \end{aligned} \tag{15}$$

$$j=3\text{:}\quad \begin{pmatrix} 2(1-i\alpha)\phi_x^2+2(-2+i\beta)u_0v_0 & (-2+i\beta)u_0^2 \\ (-2-i\beta)v_0^2 & 2(1+i\alpha)\phi_x^2+2(-2-i\beta)u_0v_0 \end{pmatrix}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}=\begin{pmatrix} U_3 \\ V_3 \end{pmatrix}, \tag{16}$$

$$\vdots$$

where

$$\begin{aligned} U_3= & \, iu_{1,t}+(1-i\alpha)u_{1,xx}+(-2+i\beta)u_1^2v_1+i\gamma u_1+iu_2\phi_t+(1-i\alpha)u_2\phi_{xx} \\ & +2(-2+i\beta)u_0u_1v_2+2(-2+i\beta)u_1u_2v_0+2(1-i\alpha)\phi_x u_{2,x}, \end{aligned} \tag{17}$$

$$\begin{aligned} V_3= & -iv_{1,t}+(1+i\alpha)v_{1,xx}+(-2-i\beta)v_1^2u_1-i\gamma v_1-iv_2\phi_t+(1+i\alpha)v_2\phi_{xx} \\ & +2(-2-i\beta)u_2v_0v_1+2(-2-i\beta)v_1v_2u_0+2(1+i\alpha)\phi_x v_{2,x}, \end{aligned} \tag{18}$$

With $u_2=v_2=0$, (15) reduces to

$$\begin{pmatrix} iu_{0,t}+(1-i\alpha)u_{0,xx}+2(-2+i\beta)u_0u_1v_1+(-2+i\beta)u_1^2v_0+i\gamma u_0 \\ -iv_{0,t}+(1+i\alpha)v_{0,xx}+2(-2-i\beta)v_0u_1v_1+(-2-i\beta)v_1^2u_0-i\gamma v_0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{19}$$

and the consistency conditions for $U_3\equiv 0$ and $V_3\equiv 0$ are

$$\begin{pmatrix} iu_{1,t}+(1-i\alpha)u_{1,xx}+(-2+i\beta)u_1^2v_1+i\gamma u_1 \\ -iv_{1,t}+(1+i\alpha)v_{1,xx}+(-2-i\beta)v_1^2u_1-i\gamma v_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{20}$$

so that u_3 and v_3 become arbitrary. Clearly we are able to obtain the consistent truncations of the Painlevé expansions by setting

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix}\equiv\begin{pmatrix} 0 \\ 0 \end{pmatrix}\quad\text{for}\quad j\geq 2, \tag{21}$$

in which case

$$\begin{aligned} u(x,t)&=u_0(x,t)\phi^{-1}(x,t)+u_1(x,t), \\ v(x,t)&=v_0(x,t)\phi^{-1}(x,t)+v_1(x,t). \end{aligned} \tag{22}$$

Equations (10), (11), (14), (19), (20) and (22) constitute a set of the auto-Bäcklund transformation.

For simplicity we choose $u_1=v_1=0$, so that (20) is satisfied automatically, while (14) and (19) reduce to

$$-iu_0\phi_t-2(1-i\alpha)\phi_x u_{0,x}-(1-i\alpha)u_0\phi_{xx}=0, \tag{23}$$

$$iv_0\phi_t-2(1+i\alpha)\phi_x v_{0,x}-(1+i\alpha)v_0\phi_{xx}=0, \tag{24}$$

$$iu_{0,t}+(1-i\alpha)u_{0,xx}+i\gamma u_0=0, \tag{25}$$

$$-iv_{0,t}+(1+i\alpha)v_{0,xx}-i\gamma v_0=0. \tag{26}$$

We next concentrate ourselves on (10), (11) and (23)–(26). Differentiating (11) gives rise to

$$\frac{u_{0,x}}{u_0} + \frac{v_{0,x}}{v_0} = 2\,\frac{\phi_{xx}}{\phi_x}, \tag{27}$$

which is then combined with (23) and (24) to yield

$$\phi_t = \frac{3(1+\alpha^2)}{\alpha}\,\phi_{xx}. \tag{28}$$

Substituting (28) into (23) and (24), we get after integration,

$$u_0(x,t) = \Xi(t)\,\phi_x(x,t)^{1-3i/2\alpha}, \tag{29}$$

$$v_0(x,t) = \Psi(t)\,\phi_x(x,t)^{1+3i/2\alpha}, \tag{30}$$

where $\Xi(t)$ and $\Psi(t)$ are the non-zero, differentiable, complex functions of t constrained by

$$\Xi(t)\,\Psi(t) = 1 \tag{31}$$

because of (11).

Further, Eqns. (25), (26) and (28), lead to

$$2\gamma\,\phi_x^2 + \frac{3+4\alpha^2}{\alpha}\,\phi_{xxx}\,\phi_x + \frac{3}{2\alpha}\,\phi_{xx}^2 = 0, \tag{32}$$

a quadratic homogeneous PDE which is possessed with the following type of solutions,

$$\phi(x,t) = \mathscr{C}(t)\,e^{\mathscr{M}(t)\cdot x} + \mathscr{D}(t), \tag{33}$$

where $\mathscr{C}(t)$ and $\mathscr{D}(t)$ are, non-zero, differentiable, complex functions of t, while calculations show that

$$\mathscr{M}^2 = -\frac{4\alpha\gamma}{9+8\alpha^2}, \tag{34}$$

which is only a real number in fact, or

$$\mathscr{M} = \pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}. \tag{35}$$

Whether or not $\mathscr{M}$ is real relies on the sign of γ. Substituting (29) and (33) into (25), and integrating it with respect to t yield

$$\mathscr{C}(t) = \pm\sqrt{\frac{9+8\alpha^2}{4\alpha(-\gamma)}}\cdot A\cdot\exp\left\{\left[i(1-i\alpha)\left(1-\frac{3i}{2\alpha}\right)\frac{4\alpha(-\gamma)}{9+8\alpha^2} - \frac{\gamma}{1-3i/2\alpha}\right]\cdot t\right\}, \tag{36}$$

where A is an arbitrary, non-zero, complex number.

So in the above analysis we have used symbolic computation to obtain an auto-Bäcklund transformation, and the following sets of exact solutions to the nonlinear Schrödinger equation with small dissipative perturbations,

$$u^{\pm}(x,t) = \frac{\Xi(t)\cdot\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot\mathscr{C}(t)\right]^{1-3i/2\alpha}\cdot\exp\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot\left(1-\frac{3i}{2\alpha}\right)\cdot x\right]}{\mathscr{C}(t)\cdot\exp\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot x\right]+\mathscr{D}(t)}, \tag{37}$$

$$v^{\pm}(x,t) = \frac{\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot\mathscr{C}(t)\right]^{1+3i/2\alpha}\cdot\exp\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot\left(1+\frac{3i}{2\alpha}\right)\cdot x\right]}{\Xi(t)\cdot\left\{\mathscr{C}(t)\cdot\exp\left[\pm\sqrt{\frac{4\alpha(-\gamma)}{9+8\alpha^2}}\cdot x\right]+\mathscr{D}(t)\right\}}, \tag{38}$$

where $\mathscr{C}(t)$ is given by (36), while $\Xi(t)$ and $\mathscr{D}(t)$ remain arbitrary.

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